

Symmetries of the Relativistic Two-Boson System in External Field

Philippe Droz-Vincent

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Abstract We investigate the survival of symmetries in a relativistic system of two mutually interacting bosons coupled with an external field, when this field is “strongly” translation invariant in some directions and additionally remains unchanged by other isometries of spacetime. Since the relativistic interactions cannot be composed additively, it is not *a priori* guaranteed that the two-body system inherits all the symmetries of the external potential. However, using an ansatz which permits to preserve the compatibility of the mass-shell constraints in the presence of the field, we show how the “surviving isometries” can actually be implemented in the two-body wave equations.

Keywords Relativistic quantum mechanics · Relativistic wave equations · Mass-shell constraints

1 Introduction, Notation

Applying an external field to a particle generally spoils Poincaré invariance. But in many cases of interest some piece of this invariance still survives, because the external field itself exhibits certain kind of symmetry; for instance a static Coulomb field applied to a charged particle preserves spherical symmetry though it breaks space translation invariance.

At least insofar as scalar particles are concerned, the symmetries of the field could be characterized as the symmetries of the one-body motion in this field because (through Noether’s theorem) they are automatically reflected in the motion of a test particle.

When external forces are applied to a system of *several* particles undergoing mutual interactions, it is tempting to expect a similar situation; in other words it would be natural to formulate a general principle of invariance under the surviving isometries, as follows:

Principle of Isometric Invariance If the external potentials applied to the system remain invariant under a transformation of the Poincaré group, then the system should enjoy the same symmetry.

P. Droz-Vincent (✉)
Observatoire de Paris, CNRS, Université Paris Diderot, 5 place Jules Janssen, 92195 Meudon, France
e-mail: philippe.droz-vincent@obspm.fr

The Galilean analog of this statement is trivial, because usually all the interactions arise additively in the non-relativistic Hamiltonian.

Insofar as the equation of motion is concerned, the relativistic dynamics of a *single* particle automatically agrees with the principle [1]¹. In contrast, as soon as $N > 1$, it is by no means obvious that N -body *relativistic* dynamics can always be constructed in agreement with the principle of isometric invariance.

Indeed relativistic interactions cannot be just linearly composed; such a complication is bound to arise in any formulation of relativistic dynamics (see for instance the work of Sokolov [2] using the “point form” of dynamics).

The main goal of this article consists in proving that, given a system of two mutually interacting particles, the coupling of this system to a large class of external fields can be actually realized in a way that satisfies this principle.

For analyzing these matters there exist many formulations of relativistic particle dynamics, but the more appropriate ones are those which make use of manifestly covariant mass-shell constraints [3, 4] [5, and references therein], [6]. In this framework the motion is generated by the (half) squared-mass operators and is governed by a system of N coupled wave equations [7–13]. In the two-body case, the relationship between this approach and the conventional methods of quantum field theory has been established [14, 15], [16]². An advantage of the constraint formalism over the Bethe-Salpeter equation is the natural elimination of the relative-time degree of freedom. Let us rather emphasize that in the context of mass-shell constraints (which admits a classical analog with Poisson brackets in a phase space) symmetries and first integrals have a clear-cut status: for example a constant of the motion is characterized by its commutation with both squared-mass operators.

For simplicity we focus on the case of two *scalar particles* which interact between themselves and are also submitted to external forces. Assuming that we explicitly know the term describing mutual interaction alone, the first problem is to write down wave equations that remain compatible when the external field is applied to the system; another requirement is obviously in order: one must retrieve the correct limits when either the mutual interaction or the external field vanishes.

In general this problem is not tractable in closed form, and the necessary requirements stated above are not sufficient for a full determination of the wave equations. Complementary information must be obtained either from the underlying quantum field theory or from reasonable assumptions of “simplicity” which would actually involve some implicit symmetry. The principle of isometric invariance provides a natural prescription for removing or at least reducing the ambiguities.

We shall concentrate on the cases where the external field is translation invariant (in a special way referred to as *strong*) along some directions of spacetime, because this situation allows for mass-shell constraints in closed form.

A first solution was given by J. Bijtebier [17] under the hypothesis that the applied field is strongly stationary along a (implicitly unique) timelike direction. We put forward a more systematic formulation which only requires that the external potential is strongly translation

¹In the same spirit we have elsewhere invoked isometric invariance in the definition of positive frequencies, in the context of curved spacetime. Indeed the motion of a test particle in General Relativity is physically a one-body problem in external gravitational field, but in this case the group of spacetime isometries is “smaller” than Poincaré.

²For applications in a constant magnetic field that work should be generalized in a way that would treat all the possible lab frames on the same footing.

invariant along one *or several* directions of spacetime [18].³ Such directions are labelled as “longitudinal” and, in the generic case, their orthogonal complement is spanned by the “transverse” ones; in this study we exclude the exceptional case where the longitudinal directions span a null manifold. This approach provides an Ansatz which permits to explicitly write down the squared-mass operators in a new representation; these operators in turn are strongly translation invariant, implying that strong-translation invariance is automatically preserved by the coupling.

Naturally, beside strong translation invariance, it may happen that the external potential remains unchanged also under some other isometries.

For instance (in suitable coordinates) a constant magnetic field not only is strongly translation invariant along the directions that span the plane (03) but also exhibits rotational symmetries in the planes (12) and (03).

The above principle would require that also these extra symmetries are preserved in the motion of *two* charged particles, even when we take their mutual interaction into account. In this situation the question arises as to know whether the squared-mass operators furnished by the ansatz actually respect these additional symmetries.

Although we mainly have in mind the case of a constant magnetic field, we present here a general treatment valid for any external field which enjoys strong translation invariance. Note that up to now the merit of the ansatz was to provide squared-mass operators that reduce to the correct limits when any of the interactions vanishes. But the ansatz will appear more satisfactory if we further prove that it respects isometric invariance.

In order to tackle this question we are thus led to consider the (continuous) isometries of spacetime that survive as symmetries of the system in the presence of an external field.

Section 2 deals with one-body motion in external fields admitting directions of strong translation invariance. In Sect. 3, after a brief sketch of the two-body problem in general, we focus on the case of two independent particles submitted only to external fields; their symmetries and invariances are discussed. Mutual interactions are introduced in Sect. 4, and concluding remarks are reserved to Sect. 5.

Greek indices take on the values 0, 1, 2, 3.

2 Symmetries in the One-Body Motion

We consider the potential created by the field, i.e. the interaction term, $G(q, p)$ which arises in the single-particle Hamiltonian equation of motion $2K\psi = m^2\psi$. The half-squared mass operator is

$$K = \frac{1}{2}p^2 + G \tag{1}$$

For instance for the charge e in an electromagnetic field, using the Lorentz gauge and the canonical commutation relations $[q^\alpha, p_\beta] = i\delta_\beta^\alpha$ we have

$$2G = -eA \cdot p - ep \cdot A + e^2 A \cdot A \tag{2}$$

Similarly, in a weak gravitational field such that $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}(x)$ we would have $2G = p_\mu h^{\mu\nu} p_\nu$.

³A misprint should be corrected between (3.4) and (3.5) of that article: read E_1^L, E_2^L instead of E_1, E_2 . This generalization of Ref. [17] is necessary in particular when the external field is purely magnetic (or electric) constant in all spacetime.

In general, any quantity which commutes with K is a constant of the motion. Any quantity which commutes with G canonically generates a transformation which leaves the external potential invariant. Because of the physical importance of linear and angular momenta we focus on the canonical transformations that correspond to the continuous *isometries of spacetime* (displacements).

The presence of G breaks the full Poincaré invariance. But it may happen that some element of the Poincaré Lie algebra \mathcal{P} still commutes with G . Let j be any element of \mathcal{P} , we call it a *momentum* and we may write

$$j = a^\alpha p_\alpha + \omega^{\mu\nu} m_{\mu\nu} \tag{3}$$

where $m_{\mu\nu} = q_\mu p_\nu - q_\nu p_\mu$, for some constant vector a^α and some constant skew-symmetric tensor $\omega^{\mu\nu}$. This terminology encompasses linear and angular momentum.

Since p^2 is a Casimir of \mathcal{P} it is clear that $[K, j]$ vanishes (and j is a constant of the single-particle motion) iff $[G, j] = 0$.

In this case j is a *conserved momentum in the one-body sector*.

j generates a canonical transformation referred to as a *surviving isometry*. Among all the surviving isometries there may be some translations: $G(q, p)$ is simply translation invariant along direction w^α when $[G, w \cdot p]$ vanishes. But among the symmetries respected by the presence of G we shall distinguish *strong translation invariance* defined as follows:

G is strongly translation invariant along direction w^α when both $[G, w \cdot q]$ and $[G, w \cdot p]$ vanish.

For instance if $a^\mu = (1, 0, 0, 0)$, we say that G is strongly stationary along direction a when both $[G, q^0]$ and $[G, p^0]$ vanish, etc.

This notion is basically defined within the one-body sector, although it will be useful essentially in two-body problems. Note also that strong translation invariance can be already considered at the classical level, in terms of Poisson brackets in the eight-dimensional one-body phase space.

The directions of strong translation invariance span the *longitudinal space* E^L . Assuming that E^L admits orthonormal frames (this case will be referred to as “generic” in contrast to the exceptional case where E^L is a null plane) we introduce the *transverse space* E^T as its orthocomplement. So the space of four-vectors is an orthogonal direct sum

$$E = E^L \oplus E^T \tag{4}$$

In terms of the projector onto E^L , say τ^α_β , we distinguish longitudinal and transverse parts of the canonical variables, say $q^\alpha_{aL}, p^\beta_{bL}$ and $q^\alpha_{aT}, p^\beta_{bT}$ respectively. More generally we define purely longitudinal (resp. transverse) quantities.

The Lie algebra of the Poincaré group gets split along the same line and we have a longitudinal subalgebra \mathcal{P}_L generated by $\tau^{\mu\alpha} p_\alpha$ and $\tau^{\mu\alpha} \tau^{\nu\beta} m_{\alpha\beta}$. It is obvious that any element of \mathcal{P}_L remains a conserved momentum and generates a surviving isometry. But it may happen that other isometries also survive the application of external field.

Example Consider a charge e submitted to a constant electromagnetic field $F_{\mu\nu}$ such that only $F_{12} = -F_{21} = F \neq 0$. The interaction term in the Hamiltonian equation of motion is

$$G = -\frac{e}{2}(q_1 p_2 - q_2 p_1)F - \frac{e^2}{8}((q_1)^2 + (q_2)^2)F^2 \tag{5}$$

This system is *strongly* translation invariant along any direction of the two-dimensional plane (03).

We may equally observe that it is invariant not only by rotation in this plane, but also by rotation in the plane (12) (the latter generated by the transverse angular momentum m_{12}).

Another constant of the motion is the pseudo-momentum $C = p + eA$, but its conservation results from invariance under the so-called “twisted translations” that *are not* spacetime isometries (in a nonrelativistic context, this concept of twisted translation was introduced by Avron et al. [19]).

In general, in the presence of strong translation invariance it is convenient to classify all the conserved isometries. To this end we split any four-vector ξ as $\xi^\mu = (\xi^A, \xi^\Gamma)$ where Latin (resp. Greek) capitals refer to the longitudinal (resp. transverse) directions. In this notation the longitudinal and transverse parts of the canonical coordinates are

$$q_L^\mu = (q^A, 0), \quad q_T^\nu = (0, q^\Gamma), \quad p_{Lv} = (p_A, 0), \quad p_{Tv} = (0, p_\Gamma)$$

For an arbitrary momentum j like in (3) the skew-symmetric tensor $\omega^{\mu\nu}$ can be written as

$$\omega^{\mu\nu} = \begin{pmatrix} \omega^{AB} & \omega^{A\Delta} \\ \omega^{\Gamma B} & \omega^{\Gamma\Delta} \end{pmatrix} \tag{6}$$

where of course $\omega^{\Gamma B} = -\omega^{B\Gamma}$. We get

$$\omega^{\mu\nu} q_\mu p_\nu = \omega^{AB} q_A p_B + \omega^{A\Gamma} q_A p_\Gamma + \omega^{\Gamma B} q_\Gamma p_B + \omega^{\Gamma\Delta} q_\Gamma p_\Delta$$

and so on. We cast (3) into the form of a unique decomposition

$$j = j_{(L)} + j_{(T)} + j_{\text{mix}} \tag{7}$$

where

$$j_{(L)} = a^A p_A + 2\omega^{AB} q_A p_B \tag{8}$$

$$j_{(T)} = a^\Gamma p_\Gamma + 2\omega^{\Gamma\Delta} q_\Gamma p_\Delta \tag{9}$$

$$j_{\text{mix}} = 2\omega^{A\Delta} q_A p_\Delta + 2\omega^{A\Delta} q_\Delta p_A \tag{10}$$

Any operator which involves only q_L and p_L (resp. q_T and p_T) is called *longitudinal* (resp. *transverse*). Beware that a longitudinal *component* of a vector is not necessarily a longitudinal *operator*.

In particular we can consider longitudinal and transverse momenta; for instance $\omega^{AB} m_{AB}$ is a longitudinal rotation, etc. The splitting (4) determines, in \mathcal{P} two remarkable subalgebras namely \mathcal{P}_L and \mathcal{P}_T formed by the longitudinal and transverse momenta respectively.

As noticed previously,

Any longitudinal momentum is a constant of the motion

although every longitudinal momentum is not necessarily the generator of a longitudinal translation. Therefore insofar as conservation is concerned the nontrivial piece, in formula (7) above, is the reduced quantity

$$j_{\text{red}} = j_{(T)} + j_{\text{mix}} \tag{11}$$

It is clear that j survives as a constant of the motion iff j_{red} does. Since it belongs to \mathcal{P} it commutes with p^2 , thus according to (1) it commutes with K iff

$$[j_{\text{red}}, G] = 0$$

In view of (9, 10) we get on the one hand

$$[j_{(T)}, G] = a^\Gamma [p_\Gamma, G] + 2\omega^{\Gamma\Delta} [q_\Gamma p_\Delta, G] \tag{12}$$

Since G is purely transverse, neither q_A nor p_B can arise in the expression of $[j_{(T)}, G]$.

On the other hand we derive from (10)

$$[j_{\text{mix}}, G] = 2\omega^{A\Delta} q_A [p_\Delta, G] + 2\omega^{\Delta A} [q_\Delta, G] p_A \tag{13}$$

But $[q_\Gamma, G]$ and also $[p_\Delta, G]$ are purely transverse; it follows that q_A and p_B arise only linearly in this expression, so $[j_{\text{mix}}, G]$ is simply linear and homogeneous with respect to the longitudinal canonical variables. Thus in order to have

$$[j_{(T)}, G] = -[j_{\text{mix}}, G]$$

both sides of this formula must vanish, which amounts to have both $j_{(T)}$ and j_{mix} separately conserved.

This situation is expressed by the conditions

$$a^\Gamma [p_\Gamma, G] + 2\omega^{\Gamma\Delta} [q_\Gamma p_\Delta, G] = 0 \tag{14}$$

$$\omega^{A\Gamma} [p_\Gamma, G] = 0 \tag{15}$$

$$\omega^{\Gamma A} [q_\Gamma, G] = 0 \tag{16}$$

Taking into account the antisymmetry of ω it is clear that the last two formulas imply the following: keep the label A fixed and consider the vector $w^\mu = (0, w^\Gamma = \omega^{A\Gamma})$. Then the quantities $w \cdot q$ and $w \cdot p$ commute with G , in other words w is a direction of strong translation invariance (unless it vanishes). But w being purely transverse this would clash with the very definition of E^T (which states that all such directions are included in E^T). And this for all A . Thus all the mixed components $\omega^{A\Gamma}$ must vanish, and no j_{mix} can be a conserved momentum. In other words

Theorem 1 *No mixed momentum can be a constant of the motion in external field.*

Corollary 1 *Any conserved momentum takes on the form $j = j_{(L)} + j_{(T)}$, where $j_{(L)}$ and $j_{(T)}$ are separately conserved.*

Example For a constant magnetic external field, with only $F_{12} \neq 0$ we have $A, B = 0, 3$ whereas $\Gamma, \Delta = 1, 2$.

$$q_L = (q^0, 0, 0, q^3), \quad q_T = (0, q^1, q^2, 0)$$

and so on.

\mathcal{P}_L is spanned by p_0, p_3, m_{03} whereas \mathcal{P}_T is spanned by p_1, p_2, m_{12} . These Lie algebras respectively obey the formulas

$$[p_0, p_3] = 0, \quad [p_0, m_{03}] = -ip_3, \quad [p_3, m_{03}] = -ip_0 \tag{17}$$

$$[p_1, p_2] = 0, \quad [p_1, m_{12}] = ip_2, \quad [p_3, m_{12}] = -ip_1 \tag{18}$$

The purely transverse quantity $j_{(T)} = m_{12}$ remains conserved.

For this example we can directly check that no mixed momentum can survive: if it were so, condition (16) would be satisfied for some choice of the coefficients $\omega^{\Gamma\Delta}$. Since the splitting of spacetime directions is $2 \oplus 2$, there are at most four independent such coefficients, say $\omega^{10}, \omega^{13}, \omega^{20}, \omega^{23}$. From (5) we derive

$$[q_1, G] = -\frac{e}{2} F_{12} q_2, \quad [q_2, G] = \frac{e}{2} F_{12} q_1$$

Inserting into (16) yields

$$\omega^{1A} q_2 - \omega^{2A} q_1 = 0$$

But the transverse canonical coordinate q^1, q^2 are independent, therefore ω^{1A} and ω^{2A} must vanish for both $A = 0$ and $A = 3$. Finally the four components of ω^{1A} and ω^{2A} are zero, which excludes the possibility that a nonvanishing j_{mix} be conserved.

3 Two-Body Motion

In the two-body sector the canonical variables are q_a, p_b submitted to the commutation relations

$$[q_a^\mu, p_{bv}] = i \delta_{ab} \delta_v^\mu$$

with $a, b, c = 1, 2$. We separate the relative variables according to

$$z^\alpha = q_1^\alpha - q_2^\alpha, \quad y^\beta = \frac{1}{2}(p_1^\beta - p_2^\beta)$$

It is convenient to set

$$Z = z^2 P^2 - (z \cdot P)^2$$

The Poincaré Lie algebra is realized in terms of the generators⁴

$$P = p_1 + p_2, \quad M = (q_1 \wedge p_1)_{\mu\nu} + (q_2 \wedge p_2)_{\mu\nu} \tag{19}$$

We can consider individual momenta j_1, j_2 where j_a depends only on q_a, p_a and set

$$J = j_1 + j_2 \tag{20}$$

⁴In the formulas concerning the two-body sector, 1, 2 are particle labels. In contradistinction, in formulas (5) and (18) devoted to the single particle, the indices 1, 2 obviously refer to spacetime directions.

so that the generator of any spacetime isometry takes on the form

$$J = a^\alpha P_\alpha + \omega^{\mu\nu} M_{\mu\nu} \quad (21)$$

In the case of two independent (*i.e.* not mutually interacting) particles, the square-mass operators are $2K_1, 2K_2$ with $K_a = K(q_a, p_a) = \frac{1}{2}p_a^2 + G_a$.

When (in addition to external coupling) the particles are mutually interacting, the individual variables cannot any more appear separately in the equations of motion. The square-mass operators are generally written as $2H_1, 2H_2$ and

$$H_a = K_a + V \quad (22)$$

where the interaction term V depends on the canonical coordinates of *both* particles. V must be chosen with care, such that $[H_1, H_2]$ vanishes and such that Poincaré invariance is restored in the limit where the external field is turned off. We define $V^{(0)}$ as the no-external-field limit of V (more generally the label (0) refers to an isolated system). Naturally $V^{(0)}$ is supposed to commute with *all* the generators of spacetime isometries. In other words, in the absence of external field, H_1, H_2 respectively reduce to $H_1^{(0)}, H_2^{(0)}$ where $H_a^{(0)} = \frac{1}{2}p_a^2 + V^{(0)}$. In practice $V^{(0)}$ is explicitly given (as a Poincaré invariant operator) and is such that $H_1^{(0)}$ commutes with $H_2^{(0)}$. For a large class of mutual interactions we can write

$$V^{(0)} = f(Z, P^2, y \cdot P) \quad (23)$$

Realistic forms of f have been derived from quantum field theory [20, see Sect. II], [21].

As soon as one assumes the presence of an external field, one has to modify $V^{(0)}$ in such a way that now H_1 commutes with H_2 . The problem is of course nonlinear and an explicit solution is available only for special classes of external potentials. This solution is not completely unique: further considerations are needed in order to remove (at least partially) the arbitrariness. For this purpose isometric invariance will be a criterium of choice.

3.1 Two Independent Particles in External Fields

In the limit where no mutual interaction is present, the two-body motion is fully determined by the external potentials. A spacetime infinitesimal isometry (generated by J) is a symmetry of the system *as a whole* when both G_1 and G_2 commute with its generator, say

$$[J, G_1] = [J, G_2] = 0$$

This isometry is a *surviving isometry*.

In this case J is a first integral for the motion of two independent particles respectively submitted to the potentials G_1, G_2 . In view of (20) it is clear that any momentum J survives as a constant of the two-body motion iff

$$[j_1, G_1] = [j_2, G_2] = 0 \quad (24)$$

in other words j_1 and j_2 respectively survive application of the external potentials G_1 and G_2 in the one-body problem.

Surviving isometries may include rotations and translations. A translation along w is a surviving isometry provided that $w \cdot P$ commutes with both potentials, which makes w at least a direction of *simple* translation invariance in the two-body sector, say

$$[G_a, w \cdot P] = 0 \quad (25)$$

or equivalently

$$[G_1, w \cdot p_1] = [G_2, w \cdot p_2] = 0 \tag{26}$$

But we shall be more specially interested by *strong translation* invariance, defined as follows by analogy with the one-body case; we say that

Definition The *couple of potentials* G_1, G_2 is strongly translation invariant along direction w when each potential separately is strongly translation invariant along w in the one-body sector, in other words

$$[G_1, w \cdot q_1] = [G_1, w \cdot p_1] = 0, \quad [G_2, w \cdot q_2] = [G_2, w \cdot p_2] = 0, \tag{27}$$

When they exist, the directions of strong translation invariance (for the two-body sector) span a linear subspace E^L included in the space of four-vectors, and the projection of any vector onto E^L is obtained with help of a tensor τ .

For distinguishable particles it may happen that G_1 and G_2 be strongly invariant along *distinct* longitudinal spaces, E_1^L, E_2^L (this situation would correspond to the existence of two distinct projectors τ_1, τ_2). Still the *common* directions of strong translation invariance span the linear space $E^L = E_1^L \cap E_2^L$ corresponding to a single projector τ . But in general E_1^L, E_2^L and E^L might be all different, which would allow the possibility of different splittings in the one-body sector and the two-body one.

For simplicity we shall focus on the simple case where $E_1^L = E_2^L = E^L$. This situation is ensured by assuming that the external couplings are of the same kind for both particles, in the following sense:

Definition The external couplings are *of the same kind for both particles* when there exists a one-body potential $G(\alpha, q, p)$ where α is a coupling parameter, such that $G_a = G(\alpha_a, q_a, p_a)$ for $a = 1, 2$, with nonvanishing coupling constants α_1, α_2 .

In other words both are submitted to the same field with possibly distinct coupling constants. The most simple example is given by two different charges if we neglect their mutual interaction in front of the external field. Our definition discards the special case where one coupling constant, say α_2 , vanishes because (if $\alpha_1 \neq 0$) it leads to $E_1^L \neq E_2^L$.

To summarize, a surviving isometry may involve rotations and translations, the latter being strong or not. In the sequel we shall assume that

- (a) *the external potential admits one or several directions of strong translation invariance*
- (b) *both couplings are of the same kind and E^L is generic (not a null plane).*

Again the space of four-vectors is split as in (4) and we define the longitudinal piece of any vector, say $\xi_L^\alpha = \tau_\beta^\alpha \xi^\beta$. Similarly we separate the longitudinal canonical variables $q_{aL}^\alpha = \tau_\beta^\alpha q_a^\beta$ and $p_{bL}^\beta = \tau_\gamma^\beta p_b^\gamma$ from the transverse ones, say

$$q_{aL}^\alpha = (q_a^A, 0), \quad q_{aT}^\alpha = (0, q_a^\Gamma), \quad p_{bL}^\beta = (p_b^B, 0), \quad p_{bT}^\beta = (0, p_b^\Delta)$$

Since $E_1^L = E_2^L = E^L$ we have

$$J_{(L)} = j_{1(L)} + j_{2(L)}, \quad J_{(T)} = j_{1(T)} + j_{2(T)} \tag{28}$$

The external potentials G_a are purely transverse operators since they are supposed to commute with q_{aL} and p_{bL} .

With help of τ and $\bar{\tau}_\beta^\alpha = \delta_\beta^\alpha - \tau_\beta^\alpha$ we define ω^{AB} , etc., as in (6).

In the two-body sector, the Lie algebra of the Poincaré group, say ${}^2\mathcal{P}$ has the generators P_ρ and $M_{\mu\nu}$. In view of (4) it has longitudinal and transverse subalgebras, say ${}^2\mathcal{P}_L$ and ${}^2\mathcal{P}_T$ respectively. For instance in the constant magnetic case ${}^2\mathcal{P}_L$ is spanned by P_0, P_3, M_{03} , with commutators analogous to those in formula (17).

Any element of the Poincaré algebra can be written as in (21), but also

$$J = a \cdot P + \omega^{AB} M_{AB} + \omega^{\Gamma\Delta} M_{\Gamma\Delta} + \omega^{A\Gamma} M_{A\Gamma} + \omega^{\Gamma B} M_{\Gamma B} \tag{29}$$

if we split the tensor ω into four pieces corresponding to purely longitudinal (resp. transverse) parts and the mixed parts, say $\omega_{AB}, \omega_{\Gamma\Delta}, \omega_{A\Delta}, \omega_{\Gamma B}$. We get

$$J = J_{(L)} + J_{(T)} + J_{\text{mix}} \tag{30}$$

with

$$J_{(L)} = a^A P_A + \omega^{AB} M_{AB} \tag{31}$$

$$J_{(T)} = a^\Gamma P_\Gamma + \omega^{\Gamma\Delta} M_{\Gamma\Delta} \tag{32}$$

$$J_{\text{mix}} = \omega^{A\Gamma} M_{A\Gamma} + \omega^{\Delta B} M_{\Delta B} = 2\omega^{A\Gamma} M_{A\Gamma} \tag{33}$$

A glance at (10) shows that

$$J_{\text{mix}} = j_{1\text{mix}} + j_{2\text{mix}} \tag{34}$$

Now, in search for the conditions which make a momentum J to be conserved in the motion of two independent particles, looking at formulas (24), (28), (30), (34) we are left with two separate problems in the one-body sector. Applying the results of the previous section we obtain an extension of Theorem 1 and Corollary 1 to the two-body sector,

Proposition 1 *For independent particles no J_{mix} can be conserved, and when J is conserved we have $J_{(L)}$ and $J_{(T)}$ separately conserved.*

So any isometry of spacetime can be decomposed as in (30). The first piece is $J_{(L)}$ which depends only on q^A and p_B thus commutes with G_1, G_2 . In other words

Proposition 2 *All purely longitudinal momenta survive as constants of the motion of two independent particles.*

In contradistinction the purely transverse momenta may fail to be conserved. For instance in the magnetic example above, P_1, P_2 are not conserved although M_{12} is.

4 Mutually Interacting Particles

The Ansatz is as follows [17, 18]. The external-field representation is formally obtained with help of e^{iB} where $B = TL$ is the commutative product of a transverse operator by a

longitudinal one, namely

$$T = y_T \cdot P_T + G_1 - G_2 \tag{35}$$

$$L = \frac{P_L \cdot z_L}{P_L^2} \tag{36}$$

The transformed square-mass operators are

$$H'_a = K'_a + V' \tag{37}$$

with

$$K'_1 + K'_2 = K_1 + K_2 - 2T \frac{y_L \cdot P_L}{P_L^2} + \frac{T^2}{P_L^2} \tag{38}$$

$$K'_1 - K'_2 = y_L \cdot P_L \tag{39}$$

$$V' = f(\widehat{Z}, P^2, y_L \cdot P_L) \tag{40}$$

where f is the function in (23) and $\widehat{Z} = e^{ib} Z e^{-ib}$ where b is the no-field limit of $B = LT$. Namely

$$\widehat{Z} = Z + 2(z_T \cdot P)(z_L \cdot P) - (z_L \cdot P)^2 \frac{P_T^2}{P^2} \tag{41}$$

Remark The formulas (38)–(41) describe the *external-field representation*. In the absence of external field this representation reduces to the usual one *only after a unitary transformation*. Indeed, according to (35), we see that T and B are not cancelled by the vanishing of G_1, G_2 .

The following statement is trivial, and can be checked by hand using the canonical commutation relations,

Proposition 3 *The quantities $y_T \cdot P_T, z_T \cdot P = z_T \cdot P_T, P_T^2$ are invariant under the transverse isometries, in other words they commute with every $J_{(T)}$.*

The quantities $z_L \cdot P = z_L \cdot P_L, y_L \cdot P_L, P_L^2$ are invariant by the longitudinal isometries, in other words they commute with every $J_{(L)}$.

In addition the transverse quantities commute with all $J_{(L)}$ and the longitudinal quantities commute with all $J_{(T)}$.

Corollary 2 *Any $J_{(L)} + J_{(T)}$ commutes with \widehat{Z} (irrespective of $[J_T, K_a]$ vanishing or not).*

Proposition 4 *If a momentum J survives as a constant of the motion of independent particles, it is not affected by the transformation generated by B .*

Proof From Proposition 1 we know that such a momentum is $J = J_{(L)} + J_{(T)}$ where both $J_{(L)}$ and $J_{(T)}$ commute with K_1, K_2 . So all we have to prove is that the change of representation generated by LT produces $J' = J$.

So first consider $J_{(L)}$, it obviously commutes with T . To prove that it commutes with B we just have to check that it also commutes with L , but in (36) it is manifest that L

is invariant by the longitudinal displacements, in other words we have $[J_{(L)}, L] = 0$, thus $[J_{(L)}, B]$ vanishes.

Now consider $J_{(T)}$, being purely transverse it commutes with L . Still we are concerned about $[J_{(T)}, T]$ where T is as in (35). In T the first term $y_T \cdot P_T$ is manifestly invariant by all translations and invariant by the transverse rotation, thus $y_T \cdot P_T$ commutes with $J_{(T)}$. The second term in T is $G_1 - G_2$, but $J_{(T)}$ is supposed to commute with K_1 and K_2 , hence also with G_1 and G_2 and finally with T . To summarize $[B, J]$ vanishes which implies that $J' = J$. \square

Theorem 2 *In the context of the ansatz, with both external couplings of the same kind, if a momentum J is a constant of the motion of two independent particles, it remains a constant of the motion in the presence of mutual interaction.*

Proof We want to prove that $[H'_a, J']$ is zero. Our assumptions mean that $[K_a, J] = 0$ or equivalently that $[K'_a, J'] = 0$. But Proposition 4 implies that $J' = J$, so we have that $[K'_a, J] = 0$. In view of (37) all we have to check now is whether in (40) all the ingredients of V' actually commute with J . Proposition 1 tells that $J = J_{(L)} + J_{(T)}$. Corollary 2 ensures that \hat{Z} commutes with J . Proposition 3 implies that P^2 and $y_L \cdot P_L$ have the same property, which achieves the proof. \square

As an example consider two charges in a constant magnetic field: the momenta P_0, P_3, M_{03}, M_{12} remain conserved in the presence of a mutual interaction defined as in (23).

5 Summary and Conclusion

We have proposed a principle of invariance which seems to be a natural requirement in the presence of external fields. Then we focused on the case of external fields admitting strong translation invariance. In a first step we checked that, in the absence of mutual interaction, the description obtained in the one-body sector can be re-phrased with the same structure in the two-body framework (at this stage all surviving isometries are easily identified and each one obviously corresponds to a conserved momentum). Then we have introduced the mutual coupling, assuming that the composition of all the interactions together is performed according to the Ansatz. And finally we have shown that, in the generic case and provided both external couplings are of the same kind, this procedure ensures that the spacetime isometries which leave the external potential invariant remain symmetries of the two-body system submitted to all the interactions. So the interacting two-body system inherits the conservation laws implied by the spacetime invariances of the external field. To summarize, the principle of isometric invariance is satisfied at least in the context of strong translations, and this result enhances our confidence in the ansatz.

In the present paper we took the view that, among other possible transformations, spacetime isometries play a preferred role, owing to the physical importance of linear and angular momenta; however, for two opposite charges in the presence of a pure magnetic or pure electric field, pseudo-momentum is conserved [22]⁵ and a possible generalisation might be relevant.

⁵Formula (3.11) of that article should be written $[C^\alpha, H_b] = 0$.

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